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THE TROPICAL YEAR AND SOLAR CALENDAR

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ABSTRACT

The expression for the length of the tropical year, based on a modern theory of the motion of the Earth, is derived. The formula valid over about 8000 years centred at the present reads in days:

$$\tau = 365.242\,189\,669\,781 - 6.161\,870 \times 10^{-6}\,T - 6.44 \times 10^{-10}\,T^2,$$

where T is the time reckoned from J2000.0 and measured in Julian centuries of 365.25 ephemeris days. A comparison of the Gregorian calendar with a perfect solar calendar suggests that the former will be adequate at least during the nearest one to two thousand years. Because of high uncertainty in the Earth's rotation it is premature at present to suggest any reform that would reach further than a few thousand years into the future. An approach to calendrical analysis relying on the summation of the length of tropical years is shown to be methodologically incorrect.

RÉSUMÉ

Une formule utilisant la théorie moderne du mouvement de la Terre pour calculer la durée de l'année tropicale est présentée. Cette formule, qui est valide pour une période de 8000 ans centrée sur le présent, se lit comme suit:

$$\tau = 365.242\,189\,669\,781 - 6.161\,870 \times 10^{-6}\,T - 6.44 \times 10^{-10}\,T^2,$$

ou T est le temps écoulé depuis J2000.0 exprimé en siècles juliens de 365.25 jours du temps des éphémérides. Une comparaison entre le calendrier grégorien et un calendrier solaire parfait suggère que le calendrier grégorien sera adéquat pour les prochains un ou deux millénaires. Il est donc trop tôt pour suggérer une révision pouvant s'appliquer à une période plus étendue du futur, car notre connaissance de la rotation terrestre contient encore un haut niveau d'incertitude. Enfin, il est

démontré qu'une tentative de réviser l'analyse des calendriers reposant sur l'addition de la durée des années tropicales est méthodiquement érronée.

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1. Introduction. Among those who have studied calendar structures, opinion seems to prevail that our Gregorian calendar, introduced in 1582, although not perfect, does not require any revision to its year length or rules of intercalation of leap days for thousands of years to come. Quite a usual approach to reach such a conclusion is a comparison of the mean length of the tropical year (taken either as constant or as linearly diminishing) with the mean length of the calendar year, combined with certain summations of both. In the present study I shall demonstrate that the approach is methodologically incorrect, in spite of the approximately correct results that it normally yields.

In his recent article Peck (1990) has analysed in detail a possible reform of the solar calendar in view of the changing length of the tropical year. Unfortunately, his ideas are based on the above mentioned incorrect approach. That author also overlooked the limited time span over which the Newcomb formula is valid and, equally importantly, he ignored the variable Earth rotation. My analysis shows that our present knowledge allows us to plan solar calendars roughly for 2–3 thousands of years into the future, and going considerably beyond this range remains pure speculation carrying little, if any, practical significance. Specifically, the rules of the Gregorian calendar, used now for over 400 years, serve their purpose very satisfactorily and will do so still for at least a thousand years or so.

2. Longitude of the Sun versus solar calendar. The natural basis for computing passing tropical years is the mean longitude of the Sun reckoned from the precessionally moving equinox (the dynamical equinox or equinox of date). Whenever the longitude reaches a multiple of 360° the mean Sun crosses the vernal equinox and a new tropical year begins. In a modern theory of the motion of Earth around the Sun, VSOP82* (Bretagnon 1982), the mean longitude of the Sun referred to the dynamical equinox is given by (see also e.g. *Connaissance des Temps* 1990)

$$L = 280^\circ 27' 59''.2146 + 129\,602\,771''.363\,29\ T + 1''.093\,241\ T^2 + 0''.000\,076\,2\ T^3, \quad (1)$$

*I have often seen the letters "VSOP" on the side of a bottle of wine and I have wondered what the letters stand for. I understand that, in the context of this paper, VSOP82 refers to the 1982 version of Bretagnon's "Variations Séculaires des Orbites Planétaires". Whether this is what is meant on wine bottles remains a mystery to me. – EDITOR where T is the uniform time measured in Julian centuries and is reckoned from the fundamental epoch J2000 (January 1.5, 2000), i.e.

$$T = \frac{JD - 2\,451\,545}{36\,525}, \tag{2}$$

with JD designating the Julian Day Number, a continuous count of days since 4713 BC, January 0.5 (of the proleptic Julian calendar). Here we shall specify the days to be of equal length of 86 400 SI seconds (this specification is usually associated with renaming of JD to JED, for Julian Ephemeris Day).

As we shall soon see, the equation (1) is entirely sufficient to meet the demands of calendrical calculus. To assess its accuracy and range of validity one may refer to the Laskar (1986) paper where coefficients of polynomials of degree 10 in T are given for the longitude of the Sun referred to a fixed equinox, L', and for the precession in longitude, p_A. The longitude referred to the equinox of date, in which we are now interested, is obtained as $L = L' + p_A$. In the same paper we read that this more accurate expression is valid (within a few arcseconds) over 10 000 years. In figure 1 we have plotted the difference between the mean longitudes of the Sun calculated from the equation (1) and from the Laskar formulation. The reader may now discover how little the VSOP82 expression errs for remote epochs. In the year 6000 ($T = 40$) equation (1) yields longitudes too small by only about $1'$, and in the year 12 000 ($T = 100$) the error is still on the order of $0\overset{\circ}{.}6$. However, further millennia bring about a rapid change. Remember that an accumulated error of $1°$ corresponds to nearly 1 extra day. Since the time-honoured formula of Newcomb differs but little from the modern equation (1), we clearly see how dangerous it is to extend its use beyond the range of 10 000 years. In fact, it is also risky to use Laskar's formulae over considerably more than 10 000 years, the range they were designed for. The following discussions will be limited to a few thousands of years in which equation (1) does not introduce any significant errors.

Through skipping of the constant term in equation (1) and dividing the numerical coefficients by $1\,296\,000''$, the number of seconds of arc in the complete revolution through 360°, we obtain a convenient expression for finding the number of rotations of the Sun around the ecliptic, i.e., the number of *tropical years* elapsed between J2000 and a given epoch (T):

$$L_\tau = 100.002\,138\,397\,6\; T + 8.435\,50 \times 10^{-7}\; T^2 + 5.88 \times 10^{-11}\; T^3. \tag{3}$$

This number is to be compared with the number of *calendar years* elapsed in the same period. In the *Julian calendar* there are simply 100 T calendar years over T centuries. For the *Gregorian calendar*, in which years are on average 365.2425 days long, we have

$$L_C = \frac{3\,652\,500}{36\,524.25}\,T \tag{4}$$

calendar years in T Julian centuries.

Now, the difference between counts of tropical (equation (3)) and calendar (equation (4)) years can be expressed in days by multiplying $L_T - L_C$ by 365.2425, the number of days in the calendar year,

$$N' = 0.031\,033\,69\,T + 3.081\,00 \times 10^{-4}\,T^2 + 2.147 \times 10^{-8}\,T^3. \tag{5}$$

The particular factor of 365.2425 seems entirely appropriate with the Gregorian calendar. However, it is easy to see that replacing it by any number between 365 and 366 would not make any significant difference, unless a calendar departs from the solar calendar by more than, say, a third of a year. The above obtained formula tells us how far the Gregorian calendar advances ahead of the astronomically exact calendar, assuming the two were aligned at J2000. For example, with $T = 20$ (i.e. at the epoch J4000) $N' = 0\overset{d}{.}74$. This can be interpreted to mean that after 2000 years from now the date of the vernal equinox (say, 20 March) has shifted by approximately 1 day backwards in the Gregorian calendar (say, to 19 March around AD 4000). Thus, around that epoch, or rather somewhat earlier – when N' has accumulated to $0\overset{d}{.}5$ – one would correct the Gregorian calendar by redefining one of the leap years to be a common year (365 days long).

The above considerations assumed a constant duration of a calendar day, equal to the ephemeris day of 86400 atomic seconds. We know, however, that the Earth rotates nonuniformly on its axis and the duration of the calendar or civil day slowly but systematically lengthens. The true number of civil days elapsed between J2000 and an epoch given by T is the count of rotations of the Earth:

$$n = \int_0^T \Omega \mathrm{d}T = 36\,525\,T - \frac{\Delta T - \Delta T_0}{86\,400},$$

where $\Omega = \Omega_0 - \omega$ is the angular speed of the Earth's rotation, Ω_0 is equal to 1 rotation per 86400 SI seconds exactly, $\omega = (\mathrm{d}\Delta T/\mathrm{d}T)$, and ΔT and ΔT_0 represent the difference between the Terrestrial Dynamical Time (or the Ephemeris Time) and the Universal Time accumulated up to the epoch T and J2000 respectively. (The term ΔT_0 is included here for completeness even though we know it to be negligibly small, about 65 s). Since the difference is conventionally expressed in seconds of time we have explicitly divided it by 86400 to convert it to days. The effect of variable diurnal rotation may be incorporated into our calendrical calculations by setting $L_C = n/365.2425$. This, of course, is equivalent to

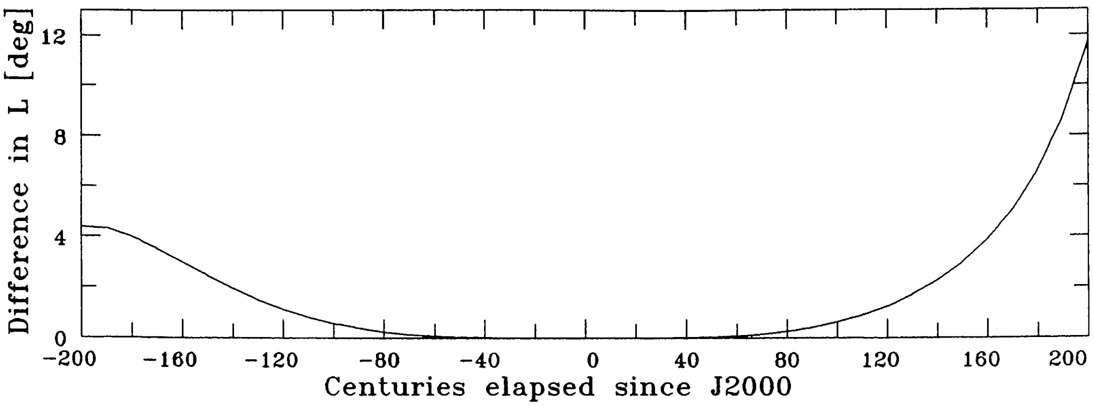

FIG. 1.—The difference between the mean longitude of the Sun defined in the French ephemeris (equation (1)) and in Laskar (1986). It shows that our third degree polynomial approximation adopted from the VSOP82 theory is very acceptable over a period of about 4000 years from the present. The longitude taken from Newcomb's theory behaves similarly to the one from the VSOP82, so much that at the scale of this figure they would not be distinguishable.

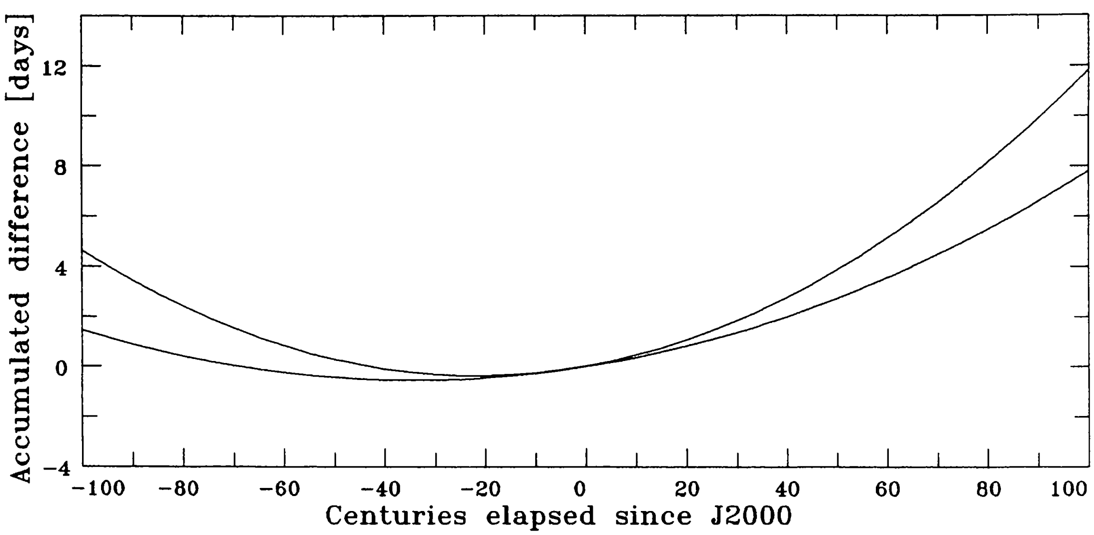

FIG. 2.—The difference between counts of tropical years and Gregorian years calculated according to equation (6). The lower and upper curves correspond to supposedly extreme scenarios of the Earth's rotation expressed by equations (7) and (8), respectively.

increasing N' by ΔT scaled to days, so that a generalized version of equation (5) becomes:

$$N = ((2.147 \times 10^{-8} T + 3.081 \times 10^{-4})T + 0.031\,033\,69)T + \frac{\Delta T - \Delta T_0}{86\,400}. \quad (6)$$

The real problem is that at present we are unable to predict accurately the duration of the day even for a moderate future, not speaking about thousands or millions of years. Besides many regular components, the rotation of our planet exhibits irregular variations on different time scales. Many researchers have attempted to fit a parabola to the measured ΔT values in order to determine the magnitude of the deceleration of the Earth's rotation. The results, when taken together, are rather discouraging. It is not unusual that formal errors of individual determinations are on the order of 1 s/cy^2, in contrast to much greater differences between obtained decelerations themselves. One of the most recent compilations of determinations of ΔT based on *telescopic observations* (McCarthy and Babcock 1986) leads to the following formula (in seconds of time)

$$\Delta T_{\mathrm{MB}} = 48.75 + 48.1699\, T + 13.3066\, T^2, \tag{7}$$

which predicts rather small values of ΔT. The coefficient at T^2 has the formal statistical uncertainty of about 0.3 s/cy^2. On the other hand, the study of *historical observations* recorded between 390 BC and AD 948, conducted also recently by Stephenson and Morrison (1984), gave an entirely different picture of the past behaviour of the Earth's rotation:

$$\Delta T_{\mathrm{SM}} = 2177 + 408.6\, T + 44.3\, T^2 \tag{8}$$

with formal error of the order of T^2 seconds.

Numerous other results generally fall between these two. Not knowing the true value of the deceleration, to proceed further in our analysis of the future calendars we may temporarily assume the Earth will rotate so that ΔT will lie generally between ΔT_{MB} and ΔT_{SM}. In that case in the year 4000 ($T = 20$, ΔT between about 6000 and 28000 seconds) we would have $0^{\mathrm{d}}\!.8 < N < 1^{\mathrm{d}}\!.1$ instead of the earlier calculated $N' = 0^{\mathrm{d}}\!.74$. This result alone entitles us to state with confidence that our *Gregorian calendar will not deviate from the exact solar calendar by significantly more than 1 day for some 2000 years to come.*

An estimate made in a similar manner for 10000 years from the present ($T = 100$; see figure 2) is much more disappointing: $8^{\mathrm{d}} < N < 12^{\mathrm{d}}$. This uncertainty of 4 days, together with quite satisfactory behaviour of the Gregorian calendar during one or two nearest millennia, renders approaches to a calendar reform both premature and unnecessary. Should our civilization survive probable social, scientific and technical revolutions of that far future it might long have forgotten about the desire to synchronize its calendar exactly with the Christian Church festivals. (Recall that the Gregorian reform aimed to fix the date of Easter in accordance with religious tradition.) If we are to predict calendar rules further than several thousand years into the future, we might also speculate that future generations may control the Earth's rotation to preserve proper agreement of a current civil calendar with astronomical phenomena.

3. The length of the tropical year. Peck (1990) observes a somewhat curious lack of an explicit formula for the length of the tropical year in the literature since the introduction of the new system of astronomical reference frames and constants. Using available astronomical literature I myself tried to locate an explanation on how to derive such an expression. Since none has been found, in the following we shall derive one.

The tropical year is an orbital period like any other of the many periods met in astronomy and associated with orbiting celestial bodies. It is essentially the reciprocal of the mean motion of the Sun. Thus, in general, if the mean longitude of the Sun can be expressed as

$$L = L_0 + aT + bT^2 + cT^3 + \cdots \tag{9}$$

with T measured, as before, in Julian centuries, then the length of the tropical year (in units of Julian centuries) can be calculated from:

$$\tau = \frac{360 \times 60 \times 60''}{\dot{L}} = \frac{1\,296\,000''}{a + 2bT + 3cT^2 + \cdots}, \tag{10}$$

where $\dot{L} = \mathrm{d}L/\mathrm{d}T$ is the rate of change (time derivative) of the longitude or centurial mean motion of the Sun. Since in practice the a term is much greater then the rest of the denominator, equation (10) can be rewritten in a more convenient form:

$$\tau = 36\,525 \times \frac{1\,296\,000}{a}\left(1 - 2\frac{b}{a}T - 3\frac{c}{a}T^2 - \cdots\right), \tag{11}$$

where the supplementary multiplier of 36 525 converts the value of τ from centuries to days, T still being in Julian centuries.

Now, if we take the longitude of the Sun from the Newcomb theory ($a = 129\,602\,768''\!.13\ \mathrm{cy}^{-1}, b = 1''\!.089\ \mathrm{cy}^{-2}, c = 0$) we easily obtain his widely known formula (which formed the basis of Peck's paper)

$$\tau_N = 365.242\,198\,79 - 6.14 \times 10^{-6}\ T' = 365.242\,192\,65 - 6.14 \times 10^{-6}\ T,$$

where $T' = T + 1$ (the original Newcomb theory is referred to the epoch J1900). At this point the reader may wish to consult the paper by Sôma and Aoki (1990) where this quantity is independently derived (the insignificant numerical differences arise from their modification of the Newcomb longitude).

Using coefficients of equation (1) in equation (11) we arrive at a more accurate and up-to-date expression for the length of the tropical year:

$$\tau = 365.242\,189\,669\,781 - 6.161\,870 \times 10^{-6}\ T - 6.44 \times 10^{-10}\ T^2. \tag{12}$$

This is the formula that replaces Newcomb's one and that Peck (1990) has unsuccessfully scanned the literature for. Note that here the days are each equal to 86 400 SI seconds; thus equation (12) is independent of the diurnal rotation of the Earth. Observe also that if we sum the length of tropical years according to this formula (or the one of Newcomb, the expression for τ_N), the accumulated error will have the magnitude similar to that displayed in figure 1, in addition to errors introduced by the method itself.

Also, it is apparent now that the meaning of equation (12) is that the *absolute length of the year is changing*. The change is certainly not due to a frictional retardation of the Earth's axial rotation, as Hope (1964) has convincingly suggested.

4. The summation of the tropical year length is incorrect. In some calendrical discussions, to find the number of tropical years in a given period the summation of the lengths of tropical years is employed. Though the numerical results will be usually acceptable this method is incorrect in principle and works only because the length changes very slowly indeed. To substantiate this statement suppose we rewrite our formula for longitude as $L = t + \epsilon(t)$, where t is the time expressed in years of constant duration (e.g. 365.25 ephemeris days). Then the difference of time measured in tropical and constant years is

$$N' = L - t = \epsilon(t), \tag{13}$$

exactly. In terms of the tropical year count we would have

$$\tau = \frac{1}{\dot{L}} = \frac{1}{1 + \dot{\epsilon}(t)} \approx 1 - \dot{\epsilon}(t) \tag{14}$$

(assuming $\dot{\epsilon} \ll 1$) and

$$N'' = \int_0^t \tau dt - t \approx \int_0^t [1 - \dot{\epsilon}(t)]dt - t = -\epsilon(t). \tag{15}$$

Thus, integrating the period, τ, i.e. summing up the length of tropical years, yields *almost* the same result, save the reversal of the sign. This conclusion is also intuitively appealing, for, if a tropical year were constant but slightly shorter, say by δ, than the calendar year, then in q calendar years there would be *approximately* $q + q\delta$ tropical years, and not $q - q\delta$ as the summation of lengths of tropical years alone would imply.

It should be borne in mind, however, that the exactness of $|N''|$ depends on the smallness of the term $\dot{\epsilon}(t)$, so that in general it is to be preferable to use directly the *mean longitude* of the Sun instead of the tropical year length.

5. *Conclusion.* The general formulae that Peck (1990) has derived and that he has found useful in practice, may be further generalized to conform to the methodologically correct approach and to the more accurate expression for the motion of the Sun. Clearly, the quantity $n(T) - 365\ L_T(T)$, *i.e.* the difference between the total number of days and the count of days contained in L_T 365-day years, represents the desired number of leap years in a solar calendar between J2000 and the epoch T. Thus, using Peck's Year Zero, AD0 ($JD = 1\,721\,058, T_0 = -730\,487/36\,525$), as the origin for counting years, we easily obtain a formula that allows us to find the number of leap years until epoch T, necessary in order that a given calendar closely agrees with the Sun's motion:

$$\begin{aligned} l &= n(T) - n(T_0) - 365\ [L_T(T) - L_T(T_0)] \\ &= 484.504 + 24.219\,5\ T - 3.079 \times 10^{-4}T^2 - 2.15 \times 10^{-8}T^3 \\ &\quad - \frac{\Delta T - \Delta T_0}{86\,400}, \end{aligned} \tag{16}$$

where ΔT_0 stands for the value of ΔT at the year numbered 0 (or at T_0). In this equation we can substitute $365.242\,2q/36\,525 + T_0$ for the argument T to convert it *approximately* to:

$$l = q(0.242\,313 - q(3.07 \times 10^{-8} + 2.15 \times 10^{-14}q)) - \frac{\Delta T - \Delta T_0}{86\,400}, \tag{17}$$

where now the argument is the year of the common era, q. Note that l can be expressed exactly as a function of q through the substitution $T = (365q + l)/36\,525 + T_0$ and then solving a cubic equation. However, such a solution is relatively complicated and the approximation we have made is very good indeed (over the range 0–12 000 of q, the departure from the exact solution nowhere exceeds 0.002 of a day).

The equation (17) can be directly compared with the equation (5) in Peck (1990):

$$l_p = 0.242\,315\,45q - 3.07 \times 10^{-8}q(q + 1). \tag{18}$$

We see that, apart from the term ΔT, there are differences which are essentially insignificant over the few thousands years over which one can rely on the mean longitude of the Sun. Subtracting the number of leap years actually introduced in a specified calendar from the equation (17) leads to another useful calendar formula analogous to the equation (3) in Peck (1990).

To sum up, in this paper we have shown or observed that:

– For comparison of a solar calendar with an exact count of tropical years the mean longitude of the Sun should be used instead of any summed lengths of tropical year.

- Though there exist more accurate expressions for the mean longitude of the Sun, the simple formula used in the VSOP82 theory (Bretagnon 1982) is entirely adequate for calendrical calculus up to a few thousand years into the future (at the epoch J12 000 it errs by somewhat more than half a degree).
- Because of large uncertainties in the length of the day it is speculation to give new calendar rules for epochs removed more than some 2–3 thousand years from the present.
- The mean length of the tropical year can be calculated according to equation (12). This expression is a modern replacement of the similar but less accurate formula of Newcomb.
- The number of tropical years passed between J2000 and any other epoch can be found using equation (3).
- To obtain the number of days by which the Gregorian calendar advances over the astronomical one, the use of equation (6) is suggested.
- The calendar formula worked out by Peck (1990), supposed to be practically useful, is improved. The corrected version is given here in the form of equation (17). This equation tells us how many leap years should there be in a solar calendar up to the year q of the common era.

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